MATH 102:107, CLASS 25 (FRI NOV 3)

(1) A barrel initially contains 2 kg of salt dissolved in 20 L of water. Water flows in at a rate of 0.5 L per minute, and well-mixed salt water solution flows out at the same rate.

(a) Write down a differential equation for $S(t)$, the amount of salt in the barrel at time t.

Solution: The differential equation is

$$\frac{dS}{dt} = (\text{Salt in}) - (\text{Salt out})$$
$$\frac{dS}{dt} = 0 - (\text{Concentration})(0.5)$$

this is because the water flowing in has no salt in it, and the water flows out at a rate of 0.5 L per minute. We can write the concentration of the salt in the water as the amount of salt divided by the volume of the barrel, i.e.

$$\text{Concentration} = \frac{S}{20}$$

so now our differential equation looks like

$$\boxed{\frac{dS}{dt} = -\frac{S}{20}(0.5) = -\frac{S}{40}}$$

(b) How many minutes will it take before there is only 1 kg of salt in the barrel?

Solution: We must find the value of t which gives $S(t) = 1$. We'll do this by first solving the differential equation to find $S(t)$. The general solution to the differential equation is $S(t) = Ce^{-t/40}$. The given initial condition is $S(0) = 2$, so this means that $C = 2$ and therefore $S(t) = 2e^{-t/40}$. Now solving $S(t) = 1$,

$$2e^{-t/40} = 1 \implies e^{-t/40} = 1/2$$
$$\implies -t/40 = \ln(1/2) = -\ln(2)$$
$$\implies t = 40\ln(2)$$

Therefore, it will take $\boxed{40\ln 2}$ minutes. (roughly 27.6 minutes)

(2) An object submerged in a pool of water has temperature T satisfying the differential equation

$$\frac{dT}{dt} = -5T + 30$$

Solution: See the scanned notes for the complete solution.

(3) Consider the differential equation $\frac{dy}{dt} = (2y+1)^{1/2}$.

(a) Is $y = \frac{t-1}{2}$ a solution?

Solution: No, it is not. Just check both sides

$$\frac{dy}{dt} \stackrel{?}{=} (2y+1)^{1/2}$$

$$\frac{1}{2} \stackrel{?}{=} \left(2\frac{t-1}{2}+1\right)^{1/2}$$

$$\frac{1}{2} \stackrel{?}{=} t^{1/2}$$

These are not equal.

(b) Is $y = \frac{t^2}{2} - \frac{1}{2}$ a solution?

Solution: It *almost* is, but no it is not! Check both sides

$$\frac{dy}{dt} \stackrel{?}{=} (2y+1)^{1/2}$$

$$t \stackrel{?}{=} \left(2\left(\frac{t^2}{2}-\frac{1}{2}\right)+1\right)^{1/2}$$

$$t \stackrel{?}{=} (t^2)^{1/2}$$

It is *almost* a solution - except that the right side is equal to $|t|$, not t itself! This is reflected in the fact that $y = \frac{t^2}{2} - \frac{1}{2}$ is decreasing for $t < 0$, i.e. its derivative is negative, however the differential equation requires that the derivative is positive everywhere because square roots are positive.

(4) In this question, we write down a differential equation to model the growth of a spherical cell. Let $r(t)$ be the radius of the cell at time t, and let $m(t)$ be the mass of the cell at time t. We assume that

$$\begin{aligned}\frac{dm}{dt} &= \text{Nutrients absorbed} - \text{Nutrients consumed}\\ &= aS - bV\end{aligned}$$

where S is the surface area, V is the volume, and a and b are constants. We also assume that mass is proportional to the volume, i.e. $m = cV$ where c is a constant. By writing S and V in terms of the radius r, write down a differential equation for r.

Solution: $S = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$, so let's first write everything in terms of r.

$$\frac{dm}{dt} = aS - bV$$

$$\frac{d(cV)}{dt} = aS - bV$$

$$\frac{d\left(c\frac{4}{3}\pi r^3\right)}{dt} = a(4\pi r^2) - b\left(\frac{4}{3}\pi r^3\right)$$

I'm going to pull out all of the constant terms first to cancel some out.

$$\frac{4}{3}\pi c\frac{d(r^3)}{dt} = 4\pi a r^2 - \frac{4}{3}\pi b r^3$$

$$c\frac{d(r^3)}{dt} = 3ar^2 - br^3$$

Now use the Chain rule

$$3cr^2\frac{dr}{dt} = 3ar^2 - br^3$$

$$\boxed{\frac{dr}{dt} = \frac{a}{c} - \frac{b}{3c}r}$$

and there is our differential equation for the radius of the cell. (This has steady state $r = \frac{3a}{b}$, which is the largest radius cells can sustain in this model.)